Unicycle and Bicycle Model for Car Collision Avoidance

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Harry Sood, Collaborator: Aakash Abhishek, Advisor: Dr. Jean-Baptiste Jeannin
Department of Aerospace Engineering, University of Michigan

1 INTRODUCTION

This project's goal is to formally verify safety regions for a car avoiding a collision with an obstacle by both braking and swerving. The system is represented in two separate models, the Unicycle Model and the Bicycle Model. The Unicycle Model features a single point located at the center of the rear axle of the car. The car's physical dimensions, length and width, are still accounted for in the safety regions however the motion is modeled with the rear axle point. The Bicycle Model builds upon the Unicycle Model by changing the rear axle point to a fixed wheel and adding a second, rotating wheel to the center of the front axle. The motion of the car is modeled in accordance to both these wheels. The significance of the Bicycle Model is that it changes the input controls to more realistic parameters. Instead of using turn radius and heading angle which is not feasibly directly regulated by a driver/controller, it uses turning angle (steering wheel) and velocity (pedals). The trajectories and safety regions for both models have been derived by hand and simulated in MATLAB. Using differential dynamic logic d$\mathcal{L}$, the Unicycle Model was verified with the KeYmaera X tactical theorem prover for both swerving-only and braking-while-swerving. The verification for the Bicycle Model, both rear-wheel-drive and front-wheel-drive, is still underway using the similar methods to the Unicycle Model.

This paper is heavily based on the work reported in [1] and [2], listed in the references and repeated below. While this paper focuses primarily on the kinematics of the Unicycle and Bicycle Model, [1] and [2] are more involved in the formal proofs for the Unicycle Model.

[1] Aakash Abhishek, Harry Sood, and Jean-Baptiste Jeannin. 2020, to appear. Formal verification of swerving maneuvers for car collision avoidance. In *2020 American Control Conference (ACC)*. IEEE.

[2] Aakash Abhishek, Harry Sood, and Jean-Baptiste Jeannin. 2020, to appear. Formal verification of braking while swerving in automobiles. In 2020 *International Conference on Hybrid Systems: Computation and Control (HSCC)*. ACM.

2 KINEMATIC UNICYCLE MODEL

2.1 Equations of Motion

In the frame of the Unicycle Model, the vehicle is assumed to be a rectangular object with its center of mass (COM) located at the center of the vehicle's rear axle for simplicity. The vehicle's motion is based on Ackermann's Steering Geometry (ASG). The COM is chosen to be at the center of the vehicle's rear axle due to its relationship with ASG, allowing the velocity vector $\boldsymbol{v}$ to be aligned with the vehicle's physical heading.

These assumptions form a simple kinematic model (Figure 1) with the following key variables:

- x and y represent the location of COM in a Cartesian frame.
- v represents the speed of COM.
- R represents the radius of turn of COM while the vehicle is swerving.
- θ represents the vehicle's heading while the vehicle is swerving.

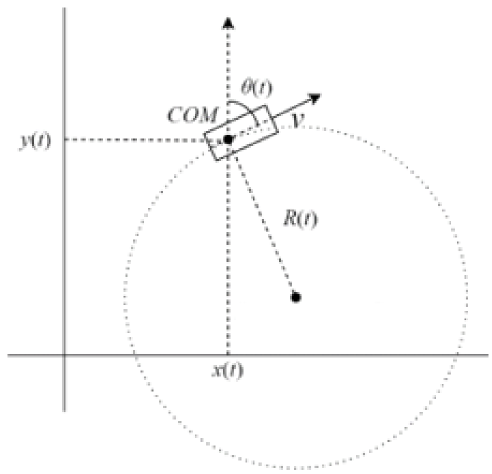

Figure 1: Kinematic Model

To account for the changing velocity incurred by braking-while-swerving, the Circle of Forces (Figure 2) is used to avoid skidding [3]. The available traction force F_{μ} is given by the circle's radius and is derived from the contact between the car's tires and the driving surface. The amount of force allocated to turning F_T and braking F_B depends on the turning angle ϕ. For instance, when $\phi = 0$ the vehicle is only braking and when $\phi = \frac{\pi}{2}$ the vehicle is only swerving.

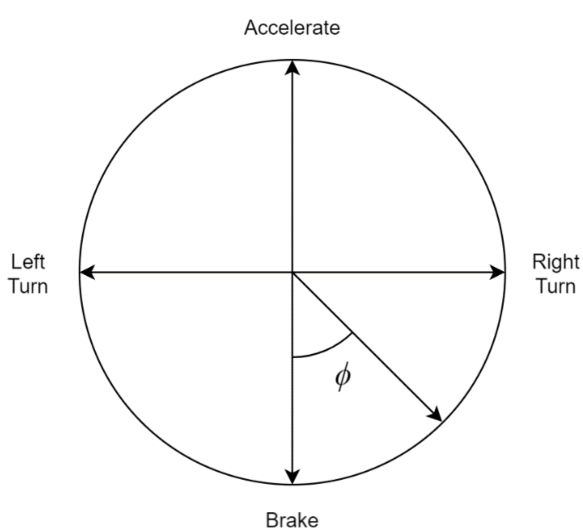

Figure 2: Circle of Forces

$$F_T = \mu mg \sin(\phi)$$
$$F_B = \mu mg \cos(\phi)$$
$$F_\mu = \mu mg = \sqrt{F_B^2 + F_T^2}$$

The equations of motion (1) – (5) are given below. The following constants have been used in the equations:

- g represents the magnitude of the acceleration due to gravity.
- μ represents the coefficient of friction between the tires and driving surface.
- μ_0 represents the coefficient of static friction, where $\mu \le \mu_0$.
- R_{min} represents the vehicle's geometric minimum turning radius.

$$\dot{x} = v(t)\sin\big(\theta(t)\big) \quad 1$$

$$\dot{y} = v(t)\cos\big(\theta(t)\big) \quad 2$$

$$\dot{v} = -\mu g \cos(\phi) \quad 3$$

$$R(t) = \begin{cases} \dfrac{v(t)^2}{\mu g \sin(\phi)}, & \text{if } \mathrm{R}_{min} \le \dfrac{v(t)^2}{\mu g \sin(\phi)} \text{ and } \mu \le \mu_0 \\ R_{min}, & \text{if } \mathrm{R}_{min} > \dfrac{v(t)^2}{\mu g \sin(\phi)} \end{cases} \quad 4$$

$$\dot{\theta}(t) = \frac{v(t)}{R(t)} = \begin{cases} \dfrac{\mu g \sin(\theta)}{v(t)}, & \text{if } \mathrm{R}_{min} \le \dfrac{v(t)^2}{\mu g \sin(\phi)} \text{ and } \mu \le \mu_0 \\ \dfrac{v(t)}{R_{min}}, & \text{if } \mathrm{R}_{min} > \dfrac{v(t)^2}{\mu g \sin(\phi)} \end{cases} \quad 5$$

2.2 Assumptions

The following assumptions have been made for the Unicycle Model, in order to simplify the system (Figure 3):

- The vehicle is approximated as a rectangle with dimensions of length L and width W.
- L is split into two segments, the distance from the rear axle to the front of the vehicle l_F and the distance from the rear axle to the rear of the vehicle l_R.
- COM is located at the center of the vehicle's rear axle.
- Any skidding between the tires and the driving surface has been ignored.

Due to the usage of ASG, a portion of the vehicle protrudes out of the trajectory at the beginning of the turn. Figure 4 shows the start of a right turn in which the point NH moves to the left. This region is referred to as a "notch" and the greatest distance of protrusion is referred to as the notch deviance dev_{NH}, given by (6). In numerical simulations, using typical dimensions of a sedan, dev_{NH} is only a few centimeters. Since the sedan's dimensions are in terms of meters, the notch can be reasonably ignored. In other words, the assumption is that assuming that the car does not have a rear segment, and $l_R = 0$ and $L = l_F$. This is also repeated in Figure 4, where the dotted line of the notch is barely separate to the solid line of the actual trajectory.

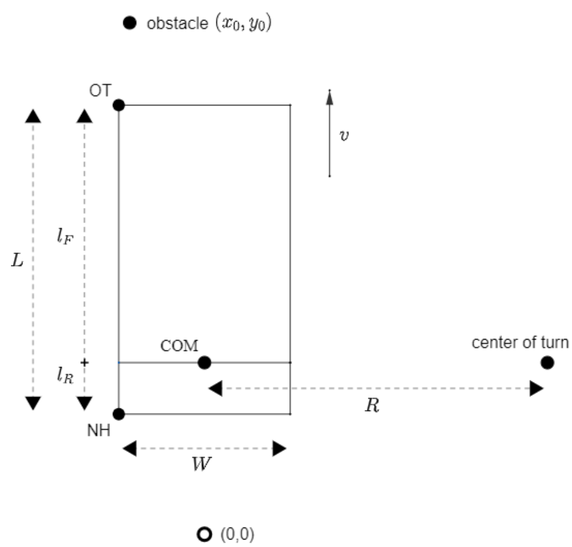

Figure 3: Assumptions

$$dev_{NH} = \sqrt{l_R^2 + \left(R + \frac{W}{2}\right)^2} - R - \frac{W}{2} \quad (6)$$

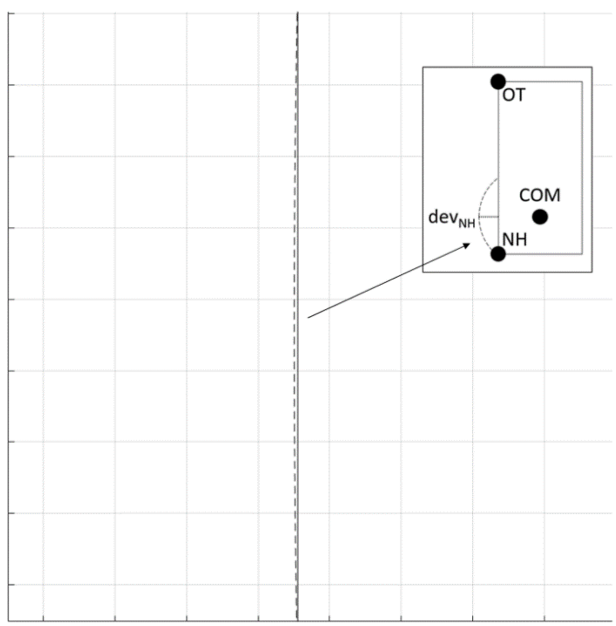

Figure 4: Notch

3 THE SWERVING-ONLY AND BRAKING-WHILE-SWERVING UNICYCLE MODEL

NOTE: While proofs for both the swerving-only and braking-while-swerving unicycle model have been completed in KeYmaera X, they will not be included in this report due to length and complexity.

3.1 Swerving-Only Unicycle Model

The collision avoidance system used in swerving-only is modeled as a discrete controller. This controller uses the vehicle's steering capabilities as an input in order to swerve into a circular trajectory followed by a straight segment upon passing the obstacle. The suggested trajectory is seen in Figure 5. The system is based on two advisory parameters, R and θ_{max}, to give the advisory in the form (R, θ_{max}). R represents the advised turning radius and θ_{max} represents the advised angle of turn. Throughout the turn, the radius must comply with the constraints $R \geq R_{min}$ and $\frac{v_0^2}{R} \leq \mu_0 g$ as posed in (4).

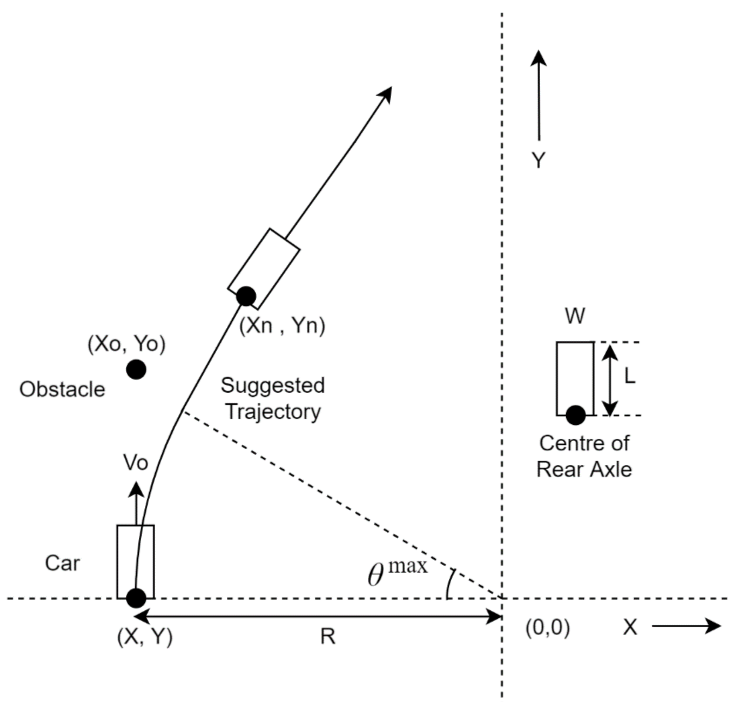

Figure 5: Collision Avoidance System

Since there is no braking for the swerving-only maneuver, it can be discerned that all force is applied towards turning. Thus, $\phi = \frac{\pi}{2}$. From this and the equations of motion (1) – (5), the solutions (7) – (8) can be found.

$$t \leq \frac{R\theta_{max}}{v_0} : \begin{cases} x(t) = -\mathrm{R}\cos\big(\theta(t)\big) \\ y(t) = \mathrm{R}\sin\big(\theta(t)\big) \\ \theta(t) = \dfrac{v_0 t}{R} \end{cases} \quad 7$$

$$t > \frac{R\theta_{max}}{v_0} : \begin{cases} x(t) = -\mathrm{R}\cos(\theta_{max}) + v_0 t \sin(\theta_{max}) \\ y(t) = \mathrm{R}\sin(\theta_{max}) + v_0 t \cos(\theta_{max}) \\ \theta(t) = \theta_{max} \end{cases} \quad 8$$

3.2 Swerving-While-Braking Unicycle Model

The braking-while-swerving maneuver uses the equations of motion (1) – (5) with the condition that ϕ can be varied anywhere between 0 and $\frac{\pi}{2}$. This is because different values of ϕ result in different combinations of braking and turning (mostly braking for $\phi < \frac{\pi}{4}$ and mostly turning for $\phi > \frac{\pi}{4}$).

Once the maneuver is initiated, ϕ can no longer be changed in this system, unless the maneuver is recalculated from the current point. This is for simplicity since it limits the system to constant braking and turning after initiation.

Allowing ϕ to be varied gives the following solutions (9) – (12) to the differential equations of motion (1) – (5).

$$c_1 = \mu g \cos(\phi)$$

$$c_2 = \mu g \sin(\phi)$$

$$v(t) = v_0 - c_1 t \quad 9$$

$$\theta(t) = \frac{c_2}{c_1} \ln\left(\frac{v_0}{v_0 - c_1 t}\right) \quad 10$$

$$x(t) = -\left\{\frac{(v_0 - c_1 t)^2 \left(2c_1 \sin\left(\frac{c_2(\ln(v_0) - \ln(v_0 - c_1 t))}{c_1}\right)\right)}{c_2^2 + 4c_1^2} + \frac{(v_0 - c_1 t)^2 \left(c_2 \cos\left(\frac{c_2(\ln(v_0) - \ln(v_0 - c_1 t))}{c_1}\right)\right)}{c_2^2 + 4c_1^2}\right\} + \frac{v_0^2 c_2}{c_2^2 + 4c_1^2} \quad 11$$

$$y(t) = -\left\{\frac{(v_0 - c_1 t)^2\left(2c_1\cos\left(\frac{c_2(\ln(v_0) - \ln(v_0 - c_1 t))}{c_1}\right)\right)}{c_2^2 + 4c_1^2} - \frac{(v_0 - c_1 t)^2\left(c_2\sin\left(\frac{c_2(\ln(v_0) - \ln(v_0 - c_1 t))}{c_1}\right)\right)}{c_2^2 + 4c_1^2}\right\} + \frac{2v_0^2 c_1}{c_2^2 + 4c_1^2} \quad 12$$

Furthermore, substituting in (7) – (8) into (9) – (10) gives the following:

$$x(t) = -\left(\frac{v^2(2c_1\sin(\theta) + c_2\cos(\theta))}{c_2^2 + 4c_1^2}\right) + \frac{v_0^2 c_2}{c_2^2 + 4c_1^2}$$

$$y(t) = -\left(\frac{v^2(2c_1\cos(\theta) - c_2\sin(\theta))}{c_2^2 + 4c_1^2}\right) + \frac{2v_0^2 c_1}{c_2^2 + 4c_1^2}$$

Numerical simulation of the above equations in MATLAB provides the car's trajectory throughout the braking-while-swerving maneuver which results in a logarithmic spiral. As shown in Figure 6, the dashed trajectory is circular and represents swerve-only while the solid trajectory is the logarithmic spiral represents braking-while-swerving. For Figure 6:

- 'x' is the instantaneous center of turn for different points during the maneuver.
- 'O' is the initial center of turn.
- (x_0, y_0) is the initial location of the car.
- F is the final location of the car, or when the car comes to a stop.

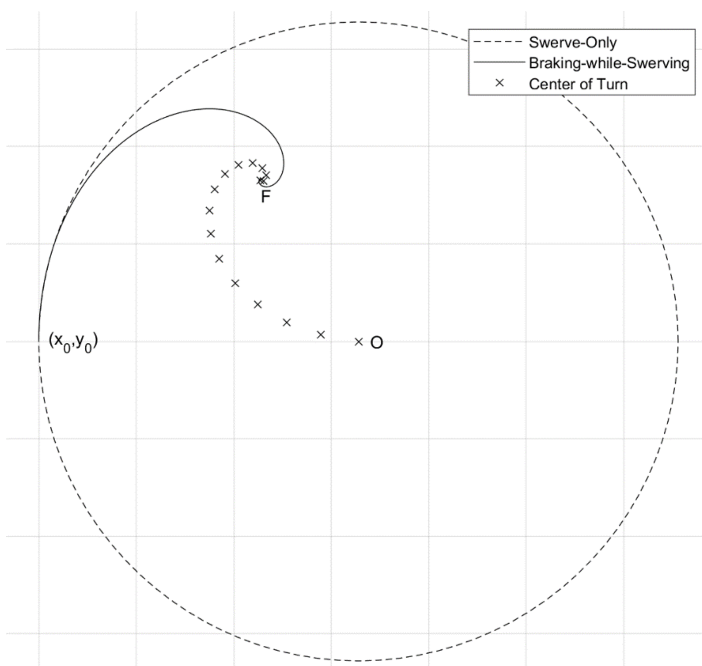

Figure 6: Swerve-Only vs Braking-while-Swerving

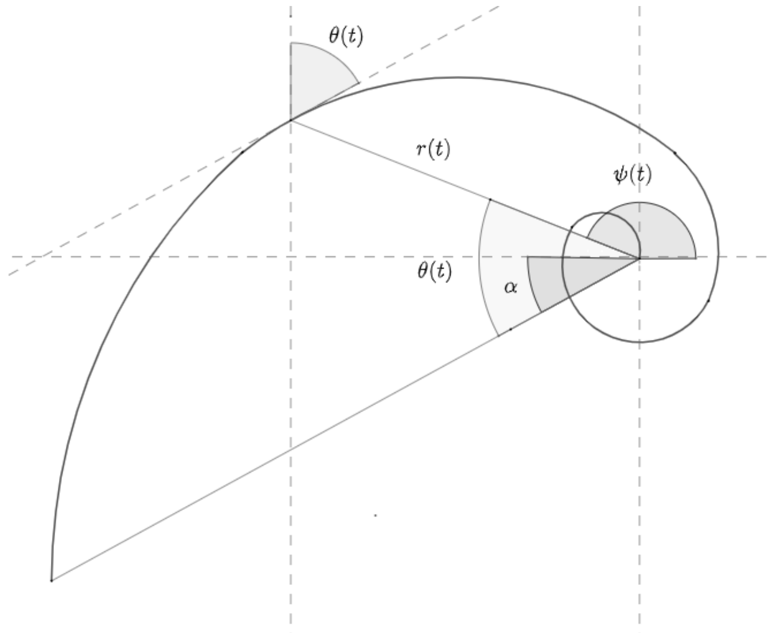

Figure 7: Spiral Trajectory with Parameters

Knowing that the car's trajectory follows a spiral, it is necessary to derive an equation in the form of a general logarithmic spiral in order to define it in terms of r and θ. The process is shown below and many key variables are depicted on Figure 7.

Below are solutions for the location of COM with respect to the final point F. F, as seen in Figure 6, is the point where the car stops. They are denoted x_F and y_F.

$$x_F(t) = \frac{-v^2(2c_1\sin(\theta) + c_2\cos(\theta)}{c_2^2 + 4c_1^2} \quad 13$$

$$y_F(t) = \frac{-v^2(2c_1\cos(\theta) - c_2\sin(\theta)}{c_2^2 + 4c_1^2} \quad 14$$

Manipulating (11) – (12), the following equation can be derived.

$$x_F(t)^2 + y_F(t)^2 = \frac{v(t)^4}{c_2^2 + 4c_1^2} = \frac{v_0^4 e^{-\left(\frac{4c_1}{c_2}\right)\theta(t)}}{c_2^2 + 4c_1^2} \quad 15$$

Since the equation for the logarithmic spiral is in the polar coordinate system, (13) – (14) must be converted.

$$x_F(t) = r(t)\cos\left(\psi(t)\right)$$

$$y_F(t) = r(t)\sin\left(\psi(t)\right)$$

Additionally, an equation for $r(t)$ can be found using (15). Following which, equations for $\psi(t)$ can be determined.

$$r(t) = \sqrt{x_F(t)^2 + y_F(t)^2} = \frac{v_0^2 e^{-\left(\frac{2c_1}{c_2}\right)\theta(t)}}{\sqrt{c_2^2 + 4c_1^2}} \tag{16}$$

$$\cos\big(\psi(t)\big) = -\frac{2c_1 \sin\big(\theta(t)\big) + c_2 \cos\big(\theta(t)\big)}{\sqrt{c_2^2 + 4c_1^2}} \tag{17}$$

$$\sin\big(\psi(t)\big) = -\frac{2c_1 \cos\big(\theta(t)\big) - c_2 \sin\big(\theta(t)\big)}{\sqrt{c_2^2 + 4c_1^2}} \tag{18}$$

Due to the fact that the equation is still needed in terms of r and θ, (17) – (18) can be used, along with a new parameter α, to relate $\theta(t)$ to $\psi(t)$.

$$\cos(\alpha) = \frac{c_2}{\sqrt{c_2^2 + 4c_1^2}}$$

$$\sin(\alpha) = \frac{2c_1}{\sqrt{c_2^2 + 4c_1^2}}$$

$$\cos\big(\psi(t)\big) = -\cos\,(\alpha - \theta(t))$$

$$\sin\big(\psi(t)\big) = -\sin\,(\alpha - \theta(t))$$

$$\theta(t) = \pi + \alpha - \psi(t)$$

Now that $\theta(t)$ is in terms of $\psi(t)$, (16) can be manipulated to be in the form of a logarithmic spiral (19). This allows the polar axis to be rotated to be $(r(t), \theta(t))$ and allows safe regions to be defined outside the spiral region (Figure 8).

$$\text{General Logarithic Spiral: } \mathrm{r} = \mathrm{k}_1 e^{k_2\theta}$$

$$r(t) = \frac{v_0^2}{\sqrt{c_2^2 + 4c_1^2}} e^{-\left(\frac{2c_1}{c_2}\right)\theta(t)} \tag{19}$$

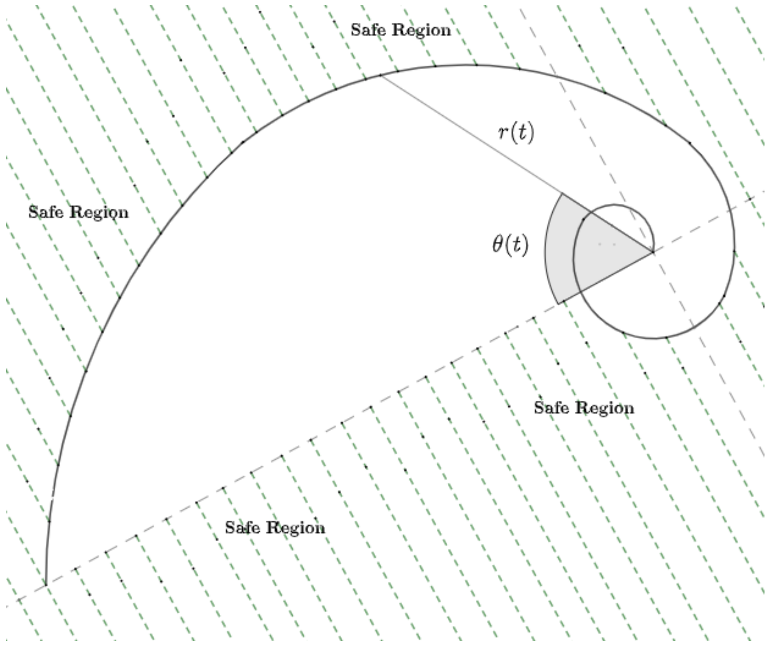

Figure 8: Spiral Trajectory with Adjusted Axis and Safe Regions

4 THE BICYCLE MODEL

4.1 Braking-While-Swerving Bicycle Model

The Bicycle Model is similar to the Unicycle Model and uses many of the same assumptions. The main difference between the two, is that the Unicycle Model follows a single point located at the center of the rear axle while the Bicycle Model follows two wheels, one at the center of the front axle and one at the center of the rear axle (Figure 9).

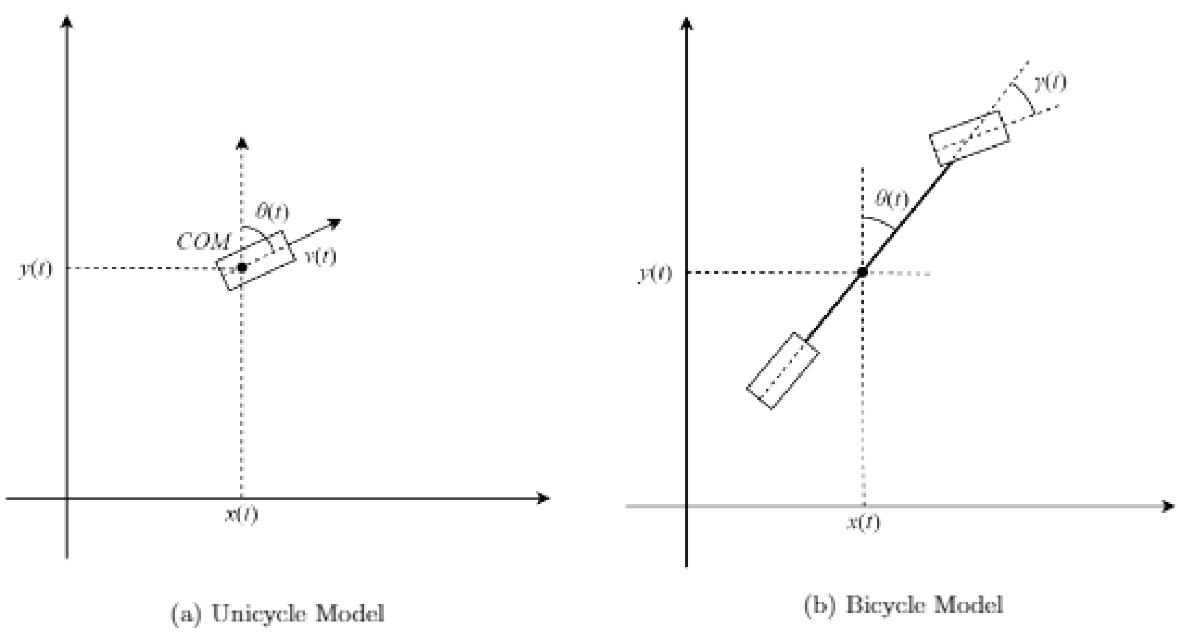

Figure 9: Spiral Trajectory with Adjusted Axis and Safe Regions

The purpose of employing the Bicycle Model is to provide more realistic advisory parameters to the controller. For instance, in the Unicycle Model the advisory is (R, θ_{max}). Both R, the turning radius, and θ_{max}, the angle of turn, are not parameters that can be directly controlled by a controller on board the vehicle. The main control capabilities on the vehicle are the steering wheel (by extension, the steering angle of the wheels), and the pedals (acceleration and by extension the velocity). The Bicycle Model uses these more realistic parameters, the steering angle and the velocity, for its advisory since they can be directly influenced by the controller.

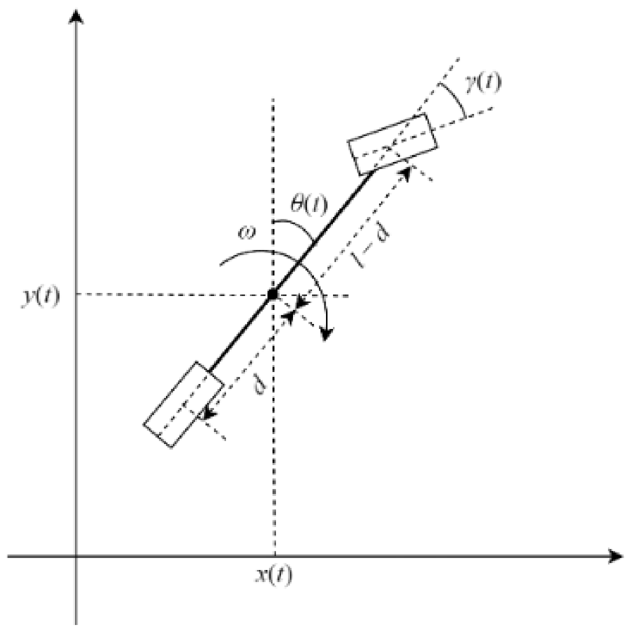

Figure 10: Bicycle Model

A stipulation of using a two-wheel system is that both rear-wheel-drive (RWD) and front-wheel-drive (FWD) must be accounted for. For both RWD and FWD, parameters from the Unicycle Model apply (unless otherwise noted) and following parameters also apply (Figure 11):

- l represents the wheelbase of the vehicle.
- d represents the distance between the rear axle and the COM of the vehicle.
- γ represents the steering angle of the vehicle.
- ω represents the rate of change of θ.

Rear-Wheel-Drive: In the RWD system, the vehicle's engine turns its rear wheels which are fixed in the orientation of the vehicle. This means that the vehicle's velocity is aligned with the rear wheels and the vehicle itself. Figure 11 depicts the RWD system.

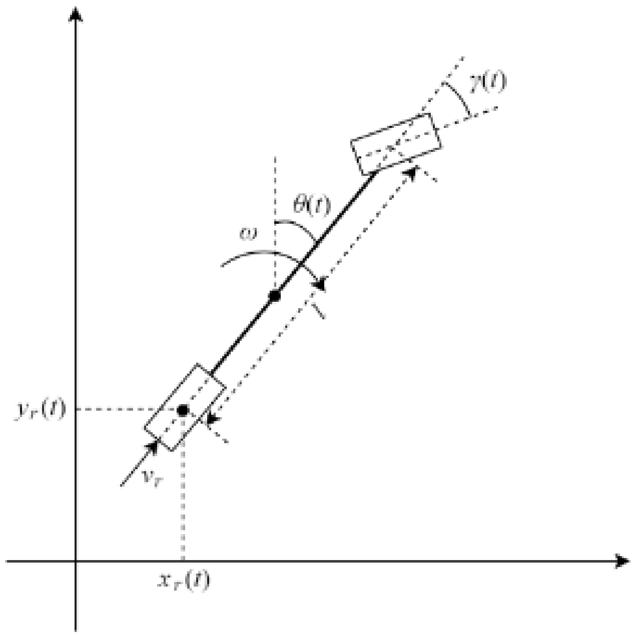

Figure 11: Rear-Wheel-Drive

The equations of motion (20) – (23) apply to the RWD system when the COM is located at the center of the front axle ($d = 0$).

$$\dot{x_r} = v_r(t)\sin\big(\theta(t)\big) \qquad 20$$

$$\dot{y_r} = v_r(t)\cos\big(\theta(t)\big) \qquad 21$$

$$\dot{\theta} = v_r(t)\tan\left(\frac{\tan(\gamma)}{l}\right) \qquad 22$$

$$\dot{v_r} = a_r \qquad 23$$

Front-Wheel-Drive: In the FWD system, the vehicle's engine turns its front wheels which are free to rotate. This means that the vehicle's velocity is aligned with the front wheels but not always the vehicle itself. Figure 12 depicts the FWD system.

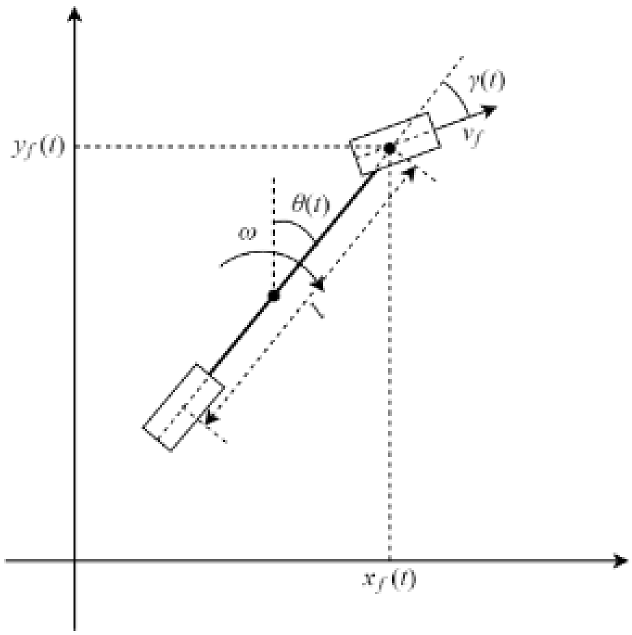

Figure 12: Front-Wheel-Drive

The equations of motion (24) – (27) apply to the FWD system when the COM is located at the center of the front axle ($d = l$).

$$\dot{x_f} = v_f(t)\sin(\theta(t) + \gamma) \quad 24$$

$$\dot{y_f} = v_f(t)\cos(\theta(t) + \gamma) \quad 25$$

$$\dot{\theta} = v_f(t)\left(\frac{\sin(\gamma)}{l}\right) \quad 26$$

$$\dot{v_f} = a_f \quad 27$$

When analyzing the trajectories of RWD and FWD, it was found that the vehicle remained on a circular path in both cases. The location of the vehicle is based on (20) – (21) and (24) – (25). In these equations, the velocity term outside the trigonometric does not affect the trajectory, only the speed the vehicle travels on the circle is affected. Of the terms inside the trigonometric functions, $\theta(t)$ changes linearly with time (as is required for a circle) and γ is constant.

Due to the circular nature of the braking-while-swerving maneuver for this model, the swerving-only maneuver does not need to be analyzed.

4.2 Proof Using Differential Dynamic Logic

Differential dynamic logic d$\mathcal{L}$ is an extension upon first-order logic that can support discrete assignments, differential equations, and choice and control loops [4]. The proof for this model is based on the proof methodology from Jean-Baptiste Jeannin's paper *A Formally Verified*

Hybrid System for Safe Advisories in the Next-Generation Airborne Collision Avoidance System [5]. For more information, consult [4] and [5].

The proof for the Bicycle Model is still underway for both rear-wheel-drive and front-wheel-drive using KeYmaera X.

5 SUMMARY AND FUTURE WORK

This paper outlines the Unicycle and Bicycle Model for a car collision avoidance system. The Unicycle Model includes both a swerving-only maneuver and a braking-while-swerving maneuver while the Bicycle Model includes a braking-while-swerving maneuver. Both maneuvers in the Unicycle Model has been formally verified while the proof for the Bicycle Model is still underway.

Direct next steps to further this research is to complete the proof for the RWD and FWD in the Bicycle Model with the braking-while-swerving maneuver. More complicated future work includes introducing uncertainty to both the Unicycle Model and Bicycle Model. Additionally, the Bicycle Model could be sophisticated to include a varying radius to form a trajectory similar to the logarithmic spiral seen in the braking-while-swerving Unicycle Model.

6 REFERENCES

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